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New Method for Extracting the Quaternion from a Rotation Matrix

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Introduction

ATTITUDE can be represented in several ways. Because the representations are of the same attitude, there must be a relationship between the different representations, and it must be possible to pass from one to another. The most popular representations of attitude are the direction cosine matrix (DCM) and the quaternion of rotation. Whereas the passage from the quaternion to the corresponding DCM is unique and straightforward, the passage from the DCM to the quaternion is not. Indeed, several algorithms were presented in the literature for computing the quaternion from the corresponding DCM[1–3]; all are based on the solution of nonlinear algebraic equations where the unknowns are the quaternion components and the knowns are the DCM elements. As noted by Shepperd,[3] Grubin's algorithm[1] degrades for large rotations and suffers from singularity when applied to a DCM that represents a 180-deg rotation. On the other hand, Klumpp's algorithm[2] is free of this singularity, but at the expense of the computation of four square roots that requires a cumbersome logic to determine the sign of the computed quaternion elements. At the present, Shepperd's algorithm or variants thereof are the simplest and most popular algorithms, but they require a square root computation and a certain voting in the way the quaternion elements are computed.

In this Note we suggest an algorithm, for extracting the quaternion from the corresponding DCM, which is valid for all attitudes and does not require any voting. Moreover, if the given DCM is not precise and, thereby, is not orthogonal, it yields the optimal quaternion in the sense that it is the quaternion that corresponds to the orthogonal matrix closest to the given imprecise DCM. The algorithm is particularly useful to users of QUEST[4] or who solve the q method directly using an algorithm that computes matrix eigenvalues and eigenvectors.

New Algorithm

In contrast to the three algorithms mentioned before that were based on the solution of nonlinear algebraic equations, the new algorithm is based on the q method.[5] The development of the new algorithm is presented in the following sections. In this section we describe the algorithm itself. The algorithm has three versions depending on the given DCM. The first two algorithms are for a given orthogonal attitude matrix.

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Version 1

1) Given an orthogonal 3×3 matrix D, form a matrix K_2 as follows:

$$K_2 = \frac{1}{2}\begin{bmatrix} d_{11} - d_{22} & d_{21} + d_{12} & d_{31} & -d_{32} \\ d_{21} + d_{12} & d_{22} - d_{11} & d_{32} & d_{31} \\ d_{31} & d_{32} & -d_{11} - d_{22} & d_{12} - d_{21} \\ -d_{32} & d_{31} & d_{12} - d_{21} & d_{11} + d_{22} \end{bmatrix} \tag{1}$$

2) Compute the eigenvector of K_2 that belongs to the eigenvalue 1. This is the sought quaternion of D.

Version 2

1) Given an orthogonal 3×3 matrix D, form a K_3 matrix as follows:

$$K_3 = \frac{1}{3}\begin{bmatrix} d_{11} - d_{22} - d_{33} & d_{21} + d_{12} & d_{31} + d_{13} & d_{23} - d_{32} \\ d_{21} + d_{12} & d_{22} - d_{11} - d_{33} & d_{32} + d_{23} & d_{31} - d_{13} \\ d_{31} + d_{13} & d_{32} + d_{23} & d_{33} - d_{11} - d_{22} & d_{12} - d_{21} \\ d_{23} - d_{32} & d_{31} - d_{13} & d_{12} - d_{21} & d_{11} + d_{22} + d_{33} \end{bmatrix} \tag{2}$$

2) Compute the eigenvector of K_3 that belongs to the eigenvalue 1. This is the sought quaternion of D.

Version 3

1) Given a nonorthogonal 3×3 matrix D, form the K_3 matrix as in Eq. (2).

2) Compute the eigenvalues of K_3.

3) Choose $\lambda_{\max}$, the largest eigenvalue of K_3.

4) Compute the eigenvector of K_3 that corresponds to the eigenvalue $\lambda_{\max}$.

This is the sought quaternion of D.

The new algorithm is based on Davenport's q method (see Refs. 5 and 6); therefore, we start our presentation of the algorithm by a short description of this method.

q Method

In 1965, Wahba[7] posed the following problem. Given are k abstract unit vectors that are resolved in a reference and in body Cartesian coordinates. Resolved in the reference coordinates, these unit vectors are denoted by $\boldsymbol{r}_i$, $i = 1, 2, \ldots, k$, and in the body coordinates they are denoted by $\boldsymbol{b}_i$, $i = 1, 2, \ldots, k$. Find the orthogonal 3×3 matrix D that minimizes the cost function L given by

$$L(D) = \frac{1}{2}\sum_{i=1}^{k} a_i |\boldsymbol{b}_i - D\boldsymbol{r}_i|^2 \tag{3}$$

where a_i is a weight we assign to the ith pair. We may want to find the quaternion rather than the matrix representation of attitude. In such a case, Eq. (3) is replaced by

$$J(\boldsymbol{q}) = \frac{1}{2}\sum_{i=1}^{k} a_i |\boldsymbol{b}_i - D(\boldsymbol{q})\boldsymbol{r}_i|^2 \tag{4}$$

In Eq. (4), we are looking for that quaternion $\boldsymbol{q}$ of unit length that minimizes J. As explained by Keat,[5] Davenport showed that the sought $\boldsymbol{q}$ is the eigenvector that corresponds to the largest eigenvalue of a certain matrix K, which is constructed as follows.

Define σ, B, S, and z

$$\sigma = \sum_{i=1}^{k} a_i \boldsymbol{b}_i^T \boldsymbol{r}_i \tag{5a}$$

$$B = \sum_{i=1}^{k} a_i \boldsymbol{b}_i \boldsymbol{r}_i^T \tag{5b}$$

$$S = B + B^T \tag{5c}$$

$$z = \sum_{i=1}^{k} a_i \boldsymbol{b}_i \times \boldsymbol{r}_i \tag{5d}$$

where T denotes the transpose, then

$$K = \left[\begin{array}{c|c} S - \sigma I_3 & z \\ \hline z^T & \sigma \end{array}\right] \tag{5e}$$

where I_3 is the third-order identity matrix. In the past, the problem of computing the eigenvalues and eigenvectors of a matrix was considered to be cumbersome; therefore, several algorithms were presented to bypass the need to solve for the eigenvalues and eigenvectors of K (Refs. 4 and 8–10). The development of such ad hoc algorithms was facilitated because K is a real symmetric matrix. The best known of these algorithms is the QUEST algorithm.[4] Usually the vectors $\boldsymbol{b}_i$ are the result of measurements, and the $\boldsymbol{r}_i$ vectors result from computations and the use of almanac; therefore, a_i, the assigned weight to each pair in Eqs. (3) and (4), represents the confidence we assign to each pair. This confidence is related, of course, to the accuracy in measuring $\boldsymbol{b}_i$ and the knowledge of the corresponding $\boldsymbol{r}_i$. If all constants add up to 1, then for error free $\boldsymbol{b}_i$ and $\boldsymbol{r}_i$, the largest eigenvalue is 1, and when the pairs contain reasonable errors, it is close to 1.

An inherent quality of an eigenvector is that if $\boldsymbol{q}$ is an eigenvector of a certain eigenvalue, then so is $\boldsymbol{q}' = -\boldsymbol{q}$. Therefore, when computing the eigenvector of the largest eigenvalue of K, one may obtain either $\boldsymbol{q}$ or its negative.[3] When used to transform vectors, either $\boldsymbol{q}$ or $-\boldsymbol{q}$ yields the same transformed vector. However, if $\boldsymbol{q}$ represents an attitude error, and to eliminate this error one wishes to command a vehicle to rotate about an Euler axis, then it is important to choose the smallest rotation about this axis. This is accomplished by choosing that quaternion whose fourth (scalar) component is positive.[3]

New Algorithm for a Precise DCM

We saw that the q method yield the quaternion that describes a rotation from one frame to another when the components of at least two vectors are known in both frames. Therefore, if we know the precise DCM that characterizes a certain rotation we can use this DCM to conveniently generate such pairs and then apply the q method to these pairs, which yield a quaternion. This is precisely the quaternion that expresses the rotation; that is, this is the quaternion that corresponds to the given DCM. Thus, we have found the sought quaternion.

Because only two vectors are necessary to determine attitude, we need just two such vector pairs. To simplify the computation we can choose the two pairs to be two unit vectors that determine two of the three coordinate axes in the reference coordinates, that is, we can choose

$$\boldsymbol{r}_1^T = [1 \quad 0 \quad 0] \tag{6a}$$

$$\boldsymbol{r}_2^T = [0 \quad 1 \quad 0] \tag{6b}$$

Let $\boldsymbol{d}_i$, $i = 1, 2, 3$, denote the three column vectors of D; then the vectors in the body system that correspond to $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$, respectively. (This is evident from the relation $\boldsymbol{b}_i = D\boldsymbol{r}_i$.) As mentioned, the largest eigenvalue of the corresponding K is equal to 1. Using $a_i = 0.5$, we can use now Eqs. (5) to construct the K matrix, and because the largest eigenvalue is known, we only have to compute its corresponding eigenvector, which is the sought quaternion. Moreover, because D and $\boldsymbol{r}_i$ are readily available, and the $\boldsymbol{r}_i$ have a simple form, we can compute K_2 directly using Eq. (1).

For computing the suitable eigenvector we can either use ad hoc routines[4,8–10] or a standard routine that computes the eigenvectors of real symmetric matrices.

Example 1, which follows, was computed using Mathcad.

$$D = \begin{bmatrix} -0.545 & 0.797 & 0.260 \\ 0.733 & 0.603 & -0.313 \\ -0.407 & 0.021 & -0.913 \end{bmatrix}, \qquad \boldsymbol{r}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{r}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{b}_1 = \begin{bmatrix} -0.545 \\ 0.733 \\ -0.407 \end{bmatrix}, \qquad \boldsymbol{b}_2 = \begin{bmatrix} 0.797 \\ 0.603 \\ 0.021 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} -0.574 & 0.765 & -0.203 & -0.010 \\ 0.765 & 0.574 & 0.010 & -0.203 \\ -0.203 & 0.010 & -0.029 & 0.032 \\ -0.010 & -0.203 & 0.032 & 0.029 \end{bmatrix}$$

The eigenvector of k_2 that corresponds to the eigenvalue 1 is the following minimizing quaternion

$$\boldsymbol{q} = \begin{bmatrix} 0.437 \\ 0.875 \\ -0.084 \\ -0.191 \end{bmatrix}$$

For a precise DCM, there is no need to use three pairs because the result will be identical.

Algorithm for an Imprecise DCM

If the given DCM is imprecise but still orthogonal, we can still use either two or three pairs and obtain the same results. However, if the given DCM is not quite orthogonal, the results will not be identical. If the two resulting quaternions are converted to DCM, the two DCMs will be orthogonal because this is an inherent quality of the expression of the DCM in terms of the corresponding quaternion. However, the quaternion, which is obtained when using three pairs, yields the DCM that is the closest orthogonal matrix (COM) to the given nonorthogonal DCM. This quality is shown next.

It has been shown[5,9] that the cost function $L(D)$, which was defined in Eq. (3), can be written as

$$L(D) = \sum_{i=1}^{k} a_i - \mathrm{tr}(DB^T) \tag{7}$$

where B is as defined in Eq. (5b). The matrix D that minimizes $L(D)$ is that matrix D that maximizes $\mathrm{tr}(DB)^T$. However, that D, which we denote by D_{orth}, is computable as follows:

$$D_{\text{orth}} = B(B^T B)^{-\frac{1}{2}} \tag{8}$$

Now this D_{orth} is precisely the COM to B, where closeness is expressed in the Euclidean norm.[11] When using three pairs, where similarly to $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$

$$\boldsymbol{r}_3^T = [0 \quad 0 \quad 1] \tag{9}$$

we obtain

$$B = \tfrac{1}{3}\boldsymbol{b}_1\mathbf{r}_1^T + \tfrac{1}{3}\boldsymbol{b}_2\boldsymbol{r}_2^T + \tfrac{1}{3}\boldsymbol{b}_3\boldsymbol{r}_3^T \tag{10}$$

With the use of the special value of the pairs $\boldsymbol{b}_1$ and $\boldsymbol{r}_1^T$, $\boldsymbol{b}_2$ and $\boldsymbol{r}_2^T$, and $\boldsymbol{b}_3$ and $\boldsymbol{r}_3^T$, it is easy to see that

$$B = \tfrac{1}{3}[\boldsymbol{d}_1 \quad \boldsymbol{d}_2 \quad \boldsymbol{d}_3] = \tfrac{1}{3}D \tag{11}$$

Therefore, Eq. (8) becomes

$$D_{\text{orth}} = D(D^T D)^{-\frac{1}{2}} \tag{12}$$

Thus, D_{orth}, which is the solution to Wahba's[7] problem, is the COM of the given DCM, D. Consequently the solution of the q method that yields the quaternion that corresponds to D_{orth} is the quaternion that yields the COM of D.

Remarks:

1) It is easy to show that when using the three, rather than the two, special pairs the expression for the K matrix in terms of the elements of D is given in Eq. (2).

2) If we use only two vectors, then from Eq. (11)

$$B = \tfrac{1}{2}[\boldsymbol{d}_1 \quad \boldsymbol{d}_2 \quad \boldsymbol{0}] \neq \tfrac{1}{2}D \tag{13}$$

Therefore, although the q method yields an optimal quaternion, it does not correspond to the COM of the given imprecise D.

3) Shepperd's[3] algorithm yields a quaternion even when D is imprecise, but it, too, does not correspond to the COM of the given imprecise D.

4) For an imprecise D, we cannot assume that the largest eigenvalue of K equals one, and so we have to find it before we compute the corresponding eigenvector.

In example 2, the given matrix D includes an error term denoted by $\mathrm{d}D$. The two matrices are as follows:

$$D = \begin{bmatrix} 0.395 & 0.362 & 0.843 \\ -0.626 & 0.796 & -0.056 \\ -0.677 & -0.498 & 0.529 \end{bmatrix}$$

$$\mathrm{d}D = \begin{bmatrix} 0.01 & 0.01 & -0.01 \\ -0.01 & 0.01 & -0.01 \\ 0.01 & 0.01 & 0.01 \end{bmatrix}$$

The three vector pairs are

$$\boldsymbol{r}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{b}_1 = \begin{bmatrix} 0.395 \\ -0.626 \\ -0.677 \end{bmatrix}, \qquad \boldsymbol{r}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\boldsymbol{b}_2 = \begin{bmatrix} 0.362 \\ 0.796 \\ -0.498 \end{bmatrix}, \qquad \boldsymbol{r}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{b}_3 = \begin{bmatrix} 0.843 \\ -0.056 \\ 0.529 \end{bmatrix}$$

The corresponding K matrix and its largest eigenvalue are

$$K_3 = \begin{bmatrix} -0.931 & -0.265 & 0.166 & 0.442 \\ -0.265 & -0.128 & -0.554 & -1.521 \\ 0.166 & -0.554 & -0.662 & 0.988 \\ 0.442 & -1.521 & 0.988 & 1.720 \end{bmatrix}, \qquad \lambda = 1.002$$

The eigenvector of K_3 that corresponds to $\lambda = 1.002$, that is, the minimizing quaternion, and its corresponding DCM are

$$\boldsymbol{q}_3 = \begin{bmatrix} 0.136 \\ -0.464 \\ 0.298 \\ 0.823 \end{bmatrix}, \qquad D_3 = \begin{bmatrix} 0.393 & 0.364 & 0.844 \\ -0.617 & 0.785 & -0.052 \\ -0.682 & -0.500 & 0.533 \end{bmatrix}$$

On the other hand, the quaternion extracted from D using Shepperd's algorithm[3] and the corresponding DCM are

$$\boldsymbol{q}_{\mathrm{Shep}} = \begin{bmatrix} 0.142 \\ -0.470 \\ 0.298 \\ 0.819 \end{bmatrix}, \qquad D_{\mathrm{Shep}} = \begin{bmatrix} 0.381 & 0.355 & 0.854 \\ -0.622 & 0.781 & -0.047 \\ -0.684 & -0.514 & 0.518 \end{bmatrix}$$

In the following we compare the COM of the given D with the DCM that corresponds to Shepperd's quaternion. We see that they are not the same. On the other hand, we see that D_3 is indeed the COM of D

$$D_{\mathrm{orth}} - D_{\mathrm{Shep}} = \begin{bmatrix} 0.012 & 0.010 & -0.009 \\ 0.006 & 0.004 & -0.005 \\ 0.002 & 0.013 & 0.015 \end{bmatrix}$$

$$D_{\mathrm{orth}} - D_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that $\boldsymbol{q}_{\mathrm{Shep}}$ is not normal; however, even if $\boldsymbol{q}_{\mathrm{Shep}}$ is normalized, the corresponding DCM is not necessarily the COM D_{orth}.

Conclusions

We presented a new algorithm for extracting the quaternion from a given DCM. The algorithm, which makes use of the q method, is simple and straightforward. We presented and discussed several variants of the algorithm. When the given DCM is indeed orthogonal, the famous K matrix of the q method is easily computed using six elements of the given DCM, and the sought quaternion is the eigenvector of K that corresponds to the eigenvalue 1. There is, of course, no need to compute the eigenvalues. Also the voting process, which exists in other algorithms for converting the DCM to its corresponding quaternion, is avoided.

If the given DCM is not quite orthogonal, then another variant of the algorithm has to be used. In this case, all nine elements of the DCM are needed to compute K. It is necessary to compute the eigenvalues of K and choose the largest one of them before computing the eigenvector of K that belongs to the largest eigenvalue. The main benefit of this algorithm variant is that the computed quaternion yields the closest orthogonal matrix to the given DCM.

The extraction of the quaternion from the K matrix can be done either using the QUEST and similar algorithms or, preferably, using a known standard algorithm for computing the eigenvalues and eigenvectors of a real symmetric matrix.

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